EE5104 Adaptive Control Systems

Part I

Dr Shuzhi Sam GE
Department of Electrical Engineering
National University of Singapore
Singapore 119260

Tel: 874 6821
Fax: 779 1103
E-mail: elegesz@nus.edu.sg
Office: E4-05-26F
http://vlab.ee.nus.edu.sg/~sge

References

1. K.J. Astrom and B. Wittenmark, Adaptive Control, Addison Wesley, 1989 (1995).
2. K.S. Narendra and A.M. Annaswamy, Stable Adaptive Systems, Prentice-Hall, 1989.
3. G.C. Goodwin and K.S. Sin, Adaptive Filtering, Prediction and Control, Prentice, 1984.

Contents

	Contents	No. of Hours
1	**Introduction** Adaptive schemes. Adaptive control theory. Applications. Situations when constant Gain feedback is insufficient. Robust control. The adaptive control problem.	3
2	**Model Reference Adaptive Systems** The model following problem. MRAS based on stability theory. Model following when the full state is measurable. Direct MRAS for general linear systems. Prior knowledge in MRAS. MRAS for partially known systems. Use of robust estimation methods in MRAS.	8
3	**Self-Tuning Regulators** The basic idea. Indirect self-tuning regulators. Direct Self-tuning regulators. Linear Quadratic STR. Adaptive Predictive control. Prior knowledge in STR.	8
4	**Real-Time Parameter Estimation** Linear-in-the-parameters model. Least squares estimation. Experimental conditions. Recursive estimators. Extended least squares. Robust estimation methods (dead zone, projection). Implementation issues.	8
5	**Auto-Tuning** PID Control. Transient response methods. Methods based on relay feedback. Relay oscillations.	3
6	**Gain Scheduling** The Principle. Design of Gain-scheduling Regulators. Nonlinear Transformations. Applications of Gain-scheduling.	3
7	**Alternatives to Adaptive Control** Situations where adaptive control should not be used. Robust high gain feedback control. Self-oscillating adaptive systems. Variable structure systems.	3

Pre-requisites for this course

- Linear Systems Theory

  1. Controllability, Observability
  2. General Observer Theory

- Digital Techniques

  1. Digital filter design (e.g. Butterworth) and implementation considerations)
  2. Sampling considerations
  3. Implementation of compensators, issues of finite word length coefficient representation, and rounding errors

The course will assume that you are already very familiar with these topics.

A simple adaptive control problem

(Narendra & Annaswamy, pp.111-114)
(Astrom & Wittenmark, pp.126-127)

Consider a system to be controlled

$$y = G_{yu}u$$

with transfer function

$$G_{yu}(s) = \frac{k_p}{s - a_p}$$

In time domain

$$(p - a_p)y(t) = k_p u(t)$$

where we use the notation

$$p \equiv \frac{d}{dt}$$

for the differential operator.

Thus

$$\dot{y}(t) = a_p y(t) + k_p u(t) \tag{1}$$

We usually refer to the system to be controlled as the “plant”.

Control Objective:

- Drive y(t) to follow some reference command trajectory $y_m(t)$

For example, $y_m(t)$ may be generated by a reference system

$$\dot{y}_m(t) = a_m y_m(t) + k_m r(t)$$

$a_m < 0$, i.e. stable reference model.

Consider a control input of the form

$$u(t) = q^* y(t) + k^* r(t)$$

Then, the plant together with control (the closed-loop system) is

$$\begin{aligned}\dot{y}(t) &= a_p y(t) + k_p \{q^* y(t) + k^* r(t)\} \\ &= (a_p + k_p q^*) y(t) + k_p k^* r(t)\end{aligned}$$

If

$$q^* = \frac{a_m - a_p}{k_p} \quad \text{and} \quad k^* = \frac{k_m}{k_p}$$

then

$$\dot{y} = a_m y + k_m r$$

and the plant together with the control will match the reference model exactly because

$$e = y_m - y\,, \quad \dot{e} = a_m e$$

PROBLEM: We do not know a_p **and** k_p **!**

Adaptive Control: Find a way to achieve this automatically.

- Estimate a_p and k_p on-line, and try to use these estimates; (Will work most of the time); and

- Use a rigorous procedure to “evolve” to the right values (Very powerfull but applies only to restricted classes).

In the course we will look at both types of approaches.

For now, let us consider the rigorous approach.

Since q^*, k^* not known, consider instead

$$u(t) = q(t)y(t) + k(t)r(t),$$ **Time-varying controller gains!**

How to choose $\{q(t), k(t), t \in [0, \infty)\}$?

Adaptive law

$$\dot{q}(t) = -\mathrm{sgn}(k_p) g_1 e(t) y(t); \qquad g_1 > 0$$

$$\dot{k}(t) = -\mathrm{sgn}(k_p) g_2 e(t) r(t); \qquad g_2 > 0$$

(Nonlinear updating mechanisms!)

Thus, adaptive controllers contain both time-varying and nonlinear dynamics!!

Because of the NLTV characteristics, analysis and design usually difficult.

For this system, consider the quadratic form (recall your 3rd year mathematics!)

$$V(e, f, y) = \frac{1}{2}\left[e^2 + |k_p|\left(g_1^{-1} f^2 + g_2^{-1} y^2\right)\right]$$

with

$$e = y - y_m, \qquad f = q - q^*, \qquad y = k - k^*$$

Re-membering that q^* and k^* are constants

$$\dot{f} = \dot{q} - \dot{q}^* = -\mathrm{sgn}(k_p) g_1 e y$$
$$\dot{y} = \dot{k} - \dot{k}^* = -\mathrm{sgn}(k_p) g_2 e r$$

Note that

Reference Model:

$$\dot{y}_m = a_m y_m + k_m r$$

Plant:

$$\begin{aligned} \dot{y} &= a_p y + k_p u = a_p y + k_p q y + k_p k r \\ &= a_p y + k_p (f + q^*) y + k_p (y + k^*) r \\ &= (a_p + k_p q^*) y + k_p f y + k_p y r + k_p k^* r \end{aligned}$$

we have

$$\dot{e} = \dot{y} - \dot{y}_m = a_m e + k_p f y + k_p y r$$

Consider derivative of quadratic form V

$$\begin{aligned} \dot{V} &= e\dot{e} + |k_p| \left[g_1^{-1} f \dot{f} + g_2^{-1} y \dot{y} \right] \\ &= a_m e^2 + k_p f e y + k_p y e r - |k_p| \left[\mathrm{sgn}(k_p) f e y + \mathrm{sgn}(k_p) y e r \right] \\ &= a_m e^2 \le 0 \end{aligned}$$

Collecting together the intermediate results:

- Quadratic form $V(e, f, y) = \frac{1}{2}\left[e^2 + |k_p| \left(g_1^{-1} f^2 + g_2^{-1} y^2 \right) \right]$

  results in $\dot{V} = a_m e^2 \le 0$

- This means that

  e^2 is bounded, f^2 is bounded, and y^2 is bounded

- $f(t) \overset{\Delta}{=} q(t) - q^*$

  $y(t) \overset{\Delta}{=} k(t) - k^*$

  \ **q** (t) and k(t) are bounded

- $e(t) \overset{\Delta}{=} y(t) - y_m(t)$

  $y_m(t)$ is a reference signal, and obviously bounded

  \ y(t) is bounded

- V(e, φ,ψ) positive definite

  $\dot{V} = a_m e^2 \leq 0$

  \ V is bounded over all time

$$\int_0^t \dot{V}\big(e(t), f(t), y(t)\big)dt = \int_0^t a_m e^2(t)dt$$

$$\Rightarrow -a_m \int_0^t e^2(t)dt = V\big(e(0), f(0), y(0)\big) - V\big(e(t), f(t), y(t)\big)$$

Since $V\left(e(t), f(t), y(t)\right)$ is always bounded,

$$\lim_{t\to\infty}\int_0^t e^2(t)dt \le c_1 \quad \text{a constant}$$

- $\dot{e} = a_m e + k_p f y + k_p y r$

  $\therefore \ \dot{e}(t)$ is bounded $\forall$t

- $$\left.\begin{array}{ll}\dot{e} & bounded \\ \int_0^\infty e^2(t)dt \le c_1 & \end{array}\right\} \Rightarrow \lim_{t\to\infty} e(t) = 0$$

  i.e. $\lim_{t\to\infty}\{y(t) - y_m(t)\} = 0$

The adaptive controller achieves the designed objective!

Rigorous analysis of adaptive control is typically as difficult.

<u>Summary</u>

Plant: $\dot{y}(t) = a_p y(t) + k_p u(t)$

Reference Model: $\dot{y}_m(t) = a_m y_m(t) + k_m r(t);\quad a_m < 0$

Control Law: $u(t) = q(t) y(t) + k(t) r(t)$, Time varying

Adaptive Law: $\dot{q}(t) = -\mathrm{sgn}(k_p) g_1 e(t) y(t)$

$$\dot{k}(t) = -\mathrm{sgn}(k_p) g_2 e(t) r(t)$$

$$e(t) = y(t) - y_m(t)$$

$q(0) = k(0) = 0$, Arbitrary starting gains

<u>Result:</u>

If the adaptive controller is applied to the plant, then all signals {y, u, θ, k} are bounded and $\lim_{t\to\infty}\left[y(t) - y_m(t)\right] = 0$

A user-friendly picture

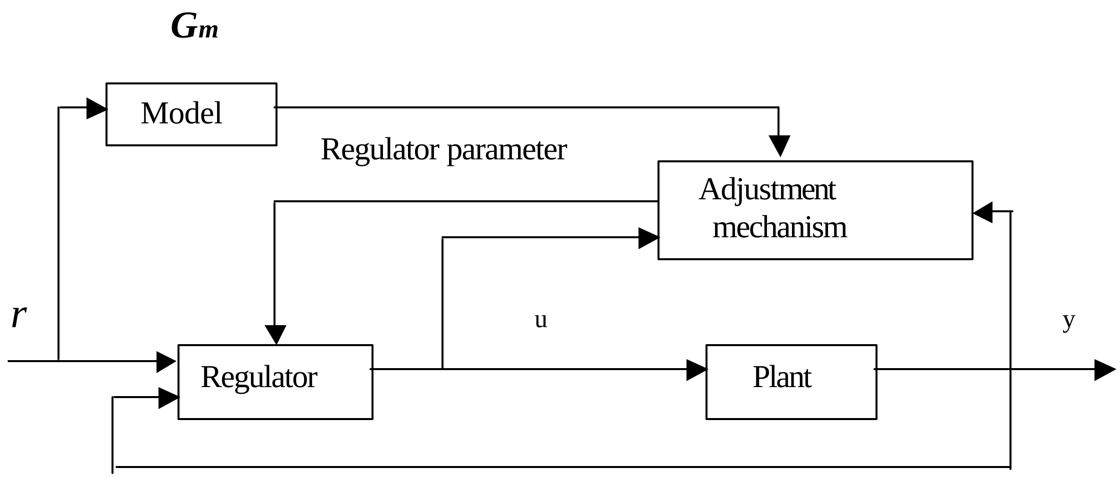


Figure 4.1 Block diagram of a model-reference adaptive system (MRAS)

Lyapunov's Direct Method

- Enables one to determine whether or not the equilibrium state of system

$$\dot{x} = f(x,t) \tag{2.1}$$

  is stable without actually determining the solution $x(t; x_0, t_0)$.

- Involves finding a suitable scalar function V(x,t) and examining its time derivative $\dot{V}(x,t)$ along the trajectory of the system.

We will only need one result from this method, which we will state without proof.

Theorem (Uniform Stability)

Assume that a scalar function V(x,t), with continuous first partial derivatives w.r.t x and t, exists and that V(x,t) satisfies the following conditions:

(i) V(x,t) is positive-definite, i.e., $\exists$ a continuous non-decreasing scalar function a such that $a(0) = 0$ and $V(x,t) \geq a(\|x\|) > 0 \quad \forall t$ and all $x \neq 0$;

(ii) V(x,t) is decrescent, i.e., ∃ a continuous non-decreasing scalar function such that $b(0) = 0$ and $b(\|x\|) \geq V(x,t)$ for all t;

(iii) $\dot{V}(x,t)$ is negative semi-definite, i.e.,

$$\dot{V}(x,t) = \frac{\partial V}{\partial t} + (\nabla V)^T f(x,t) \leq 0$$

(iv) V(x,t) is radial unbounded, i.e.,
$a(\|x\|) \to \infty$ as $\|x\| \to \infty$

then

(a) (i) and (iii) imply that the origin of differential equation (2-1) is stable;

(b) (i), (ii) and (iii) imply the origin of d.e.(2-1) is uniformly stable;

(c) (i) - (iv) imply that the origin of d.e. (2-1) is uniformly stable in the large.

LTI System and Lyapunov Stability

Consider

$$\dot{x} = Ax; \qquad x(t_0) = x_0$$

Theorem 2-2 (Narendra ….)

All solutions of the above equation tend to zero as $t \to \infty$ *if and only if all the eigenvalues of* A *have negative real parts,*

Such a matrix A is referred to as an asympototically stable matrix.

Theorem 2-10 (Narendra, …)

The equilibrium state x=0 of the LTI system

$$\dot{x} = Ax$$

is asymptotically stable if and only if, given any symmetric positive definite matrix Q*, there exists a symmetric positive definite matrix* P *which is the unique solution of the* $\frac{n(n+1)}{2}$ *linear equations*

$$A^T P + PA = -Q$$

Proof: *See Narendra & Annaswamy pp.60.*

Stability theory, in general, is a very wide field.

- See Narendra &Annaswamy, Chap.2 for a more comprehensive overview.

- We have only summarized the concepts and results that we need for adaptive control.

Adaptive control of plants with all state-variables measurable

(A similar treatment can be found in Narendra and Annaswamy pp.128 ff. However, note that some aspects are different.)

Assume that the plant is described by

$$\dot{x}_p = A_p x_p + gbu$$
$$x_p \in \Re^n , \quad u \in \Re^1$$

A_p is $(n \times n)$, b is $(n \times 1)$

b is known

This is possible in many situations.

Example 2.1: **b is known**

$$y = G_{yu} u$$

$$G_{yu}(s) = \frac{g}{s^2 + a_1 s + a_2}$$

Assume that position and velocity measurements are available.

Then, a suitable state-variable description is

$$\begin{aligned} x_1 &= y \\ \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \ddot{y} = -a_1 x_2 - a_2 x_1 + gu \end{aligned}$$

or

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -a_2 & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + g \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

Thus $b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is known exactly.

Consider plant:

$$\dot{x}_p = A_p x_p + gbu \tag{1}$$

Non-adaptive case:

$$u = q_x^{*T} x_p + q_r^* r \tag{2}$$

$$q_x^* \overset{\Delta}{=} \begin{bmatrix} q_1^* & q_2^* & \cdots & q_n^* \end{bmatrix}^T$$

Then

$$\dot{x}_p = \left[A_p + gbq_x^{*T} \right] x_p + b\left(gq_r^*\right) r$$

Assume that for these * values, the feedback control (2) achieves model matching in the sense that

$$\left.\begin{aligned} A_p + gbq_x^{*T} &\equiv A_m && (n \times n) \\ gq_r^* &\equiv g_m && scalar \end{aligned}\right\} \quad \textbf{Matching conditions}$$

Thus, control gains $q_x^* \in \Re^n$, and $q_r^* \in \Re$ exist to guarantee that the closed-loop system match the reference model

$$\dot{x}_m = A_m x_m + g_m br$$

Example 2-1 (continued)

Choice of reference model

For example, the following might be desirable

$$y_m = G_{y_m r} r$$

$$G_{y_m r}(s) = \frac{w_n^2}{s^2 + 2xw_n s + w_n^2}$$

- unity steady-state gain
- speed of response defined by natural frequency w_n
- damping specified by damping coefficient ξ

State-variable form

$$\begin{bmatrix} \dot{x}_{1m} \\ \dot{x}_{2m} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -w_n^2 & -2xw_n \end{bmatrix} \begin{bmatrix} x_{1m} \\ x_{2m} \end{bmatrix} + w_n^2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} r$$

Step Response

Title:
a:step.eps
Creator:
MATLAB, The Mathworks, Inc.
Preview:
This EPS picture was not saved
with a preview included in it.
Comment:
This EPS picture will print to a
PostScript printer, but not to
other types of printers.

Impulse Response

Title:
a:impulse.eps
Creator:
MATLAB, The Mathworks, Inc.
Preview:
This EPS picture was not saved
with a preview included in it.
Comment:
This EPS picture will print to a
PostScript printer, but not to
other types of printers.

Adaptive Case:

To adaptive match the reference model, consider the control law

$$u(t) = q_x^T(t) x_p(t) + q_r(t) r(t)$$

Define parameter errors

$$f_x(t) \overset{\Delta}{=} q_x(t) - q_x^*$$
$$f_r(t) \overset{\Delta}{=} q_r(t) - q_r^*$$

Then, the control law applied to the plant results in

$$\begin{aligned} \dot{x}_p &= A_p x_p + gb\{q_x^T x_p + q_r r\} \\ &= \left[A_p + gbq_x^{*T}\right] x_p + gbf_x^T x_p + gbq_r r \\ &= A_m x_p + gbf_x^T x_p + gbq_r r \end{aligned}$$

Compared with

$$\begin{aligned} \dot{x}_m &= A_m x_m + g_m br \\ &= A_m x_m + gbq_r^* r \end{aligned}$$

where $g_m = gq_r^*$.

Consider state error

$$e \overset{\Delta}{=} x_p - x_m$$

Then

$$\begin{aligned}\dot{e} &= A_m e + gbf_x^T x_p + gbf_r r \\ &= A_m e + gbf^T x\end{aligned}$$

where

$$f \overset{\Delta}{=} \begin{bmatrix} f_x \\ f_r \end{bmatrix}; \qquad x \overset{\Delta}{=} \begin{bmatrix} x_p \\ r \end{bmatrix}$$

For a reference model, A_m must be chosen to be a stable matrix.

Thus, it satisfies the Lyapunov equation.

$$A_m^T P + PA_m = -Q$$

i.e., for any symmetric positive definite matrix Q, there exists a symmetric positive definite P satisfying the above equation.

(From Linear Systems theory, or see Narendra & Annaswamy pp.60)

Consider a Lyapunov function candidate

$$V\left(e(t), f(t)\right) = e(t)^T Pe(t) + |g| f(t)^T \Gamma^{-1} f(t)$$

where

Γ is a symmetric positive definite (s.p.d.) matrix

We will sometime write $V(e(t), f(t))$ as V(t) for short.

Evaluate $\dot{V}$ along the trajectory of the system

$$\begin{aligned}
\dot{V} &= 2e^T P\dot{e} + 2|g| f^T \Gamma^{-1} \dot{f} \\
&= 2e^T P\left\{A_m e + gbf^T x\right\} + 2|g| f^T \Gamma^{-1} \dot{f} \\
&= 2e^T PA_m e + 2ge^T Pbf^T x + 2|g| f^T \Gamma^{-1} \dot{f} \\
&= e^T \left(A_m^T P^T + PA_m\right) e + 2ge^T Pbf^T x + 2|g| f^T \Gamma^{-1} \dot{f} \\
&= -e^T Qe + 2ge^T Pbf^T x + 2|g| f^T \Gamma^{-1} \dot{f} \qquad (3)
\end{aligned}$$

- $-e^T Qe$ term always ≤ 0

- Use the design

$$\dot{\theta} = \begin{bmatrix} \dot{\theta}_x \\ \dot{\theta}_r \end{bmatrix} = -\operatorname{sgn}(g)\Gamma x e^T Pb$$

and note that

$$\phi = \theta - \theta^*, \quad \Rightarrow \quad \dot{\phi} = \dot{\theta}$$

Equation (3) becomes

$$\begin{aligned} \dot{V} &= -e^T Qe + 2ge^T Pb\phi^T x - 2|g|\operatorname{sgn}(g)\phi^T xe^T Pb \\ &= -e^T Qe \leq 0 \end{aligned}$$

We can go through a similar analysis to show that

- $V(t)$ is positive definite and $\dot{V}(t) \leq 0$ Þ $V(t)$ is bounded
- $\|e\|, \|f\|$ $\left(hence \quad \|q\|\right)$ are bounded
- $\dot{e}$ is bounded
- $\int_0^\infty e^T Q e dt \leq c_1$
- $\lim_{t\to\infty} \|e\| = 0$ (or $\lim_{t\to\infty} \|x_p - x_m\| = 0$)

Summary

Plant

$$\dot{x}_p = A_p x_p + gbu$$

with $x_p \in \Re^n$ measurable, and b known

Matching Conditions

$$A_p + gbq_x^{*T} = A_m$$

$$gq_r^* = g_m$$

Reference Model

$$\dot{x}_m = A_m x_m + g_m br$$

Control Law

$$u(t) = q_x^T(t)x_p(t) + q_r(t)r(t)$$

Adaptive Law

$$e = x_p - x_m$$

$A_m^T P + PA_m = -Q$ *Choose* Q s.p.d., *Calculate* P s.p.d.

$$\begin{bmatrix} \dot{q}_x \\ \dot{q}_r \end{bmatrix} = -\mathrm{sgn}(g)\Gamma \begin{bmatrix} x_p \\ r \end{bmatrix} e^T Pb$$

Result: All signals $\{x_p, q_x, q_r\}$ are bounded, and $\lim_{t \to \infty} \|x_p - x_m\| = 0$

Exercise 1

Consider the plant $y = G_{yu} u$

where

$$G_{yu}(s) = \frac{g}{s^2 + a_1 s + a_2}$$

with $a_1 = 0,$ $a_2 = 1,$ y and $\dot{y}$ measurable.

Design an adaptive controller to match the reference model

$$y_m = G_{y_m r} r$$

$$G_{y_m r}(s) = \frac{w_n^2}{s^2 + 2x w_n s + w_n^2}$$

with $w_n = 2$ rad/s and $x = 0.9$.

Simulate and verify your design using MATLAB.

Incorporation of integral control with all state variables measurable

Plant:

$$\dot{x}_p = A_p x_p + gbu$$
$$y = x_1 = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} x_p$$

Matching condition:

$$A_p + gb\theta_x^{*T} = A_m$$

Integral control can be incorporated easily. Consider an additional state

$$\dot{x}_I = y - r$$

r(t) is the reference signal and $x_I = \int_0^t (y - r)dt$

Then, augmented plant

$$\begin{bmatrix} \dot{x}_p \\ \dot{x}_I \end{bmatrix} = \begin{bmatrix} A_p & \mathbf{0} \\ 1\ 0 \ldots 0 & 0 \end{bmatrix} \begin{bmatrix} x_p \\ x_I \end{bmatrix} + g \begin{bmatrix} b \\ 0 \end{bmatrix} u + \begin{bmatrix} \mathbf{0} \\ -1 \end{bmatrix} r$$

Consider

$$u = \boldsymbol{q}_x^{*T} x_p + \boldsymbol{q}_I^{*} x_I$$

The closed-loop system is

$$\begin{bmatrix} \dot{x}_p \\ \dot{x}_I \end{bmatrix} = \begin{bmatrix} A_p + gb\boldsymbol{q}_x^{*T} & gb\boldsymbol{q}_I^{*} \\ 1 \quad 0 \ \ldots \ 0 & 0 \end{bmatrix} \begin{bmatrix} x_p \\ x_I \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ -1 \end{bmatrix} r$$

Thus, an additional matching condition is

$$\begin{bmatrix} A_p + gb\boldsymbol{q}_x^{*T} & gb\boldsymbol{q}_I^{*} \\ 1 \quad 0 \ \ldots \ 0 & 0 \end{bmatrix} = \overline{A}_m$$

and

$$\dot{\overline{x}}_m = \overline{A}_m \overline{x}_m + \begin{bmatrix} \mathbf{0} \\ -1 \end{bmatrix} r$$

For the corresponding adaptive controller, consider

$$u(t) = q_x(t)^T x_p(t) + q_I(t) x_I(t)$$

and a reference model

$$\dot{\bar{x}}_m = \bar{A}_m \bar{x}_m + \begin{bmatrix} \mathbf{0} \\ -1 \end{bmatrix} r; \qquad \bar{x}_m \in \Re^{n+1}$$

Define

$$\bar{x}_p \overset{\Delta}{=} \begin{bmatrix} x_p \\ \dots \\ x_I \end{bmatrix}$$

For $\bar{e} = \bar{x}_p - \bar{x}_m$, we have

$$\begin{aligned} \dot{\bar{e}} &= \bar{A}_m \bar{e} + g \begin{bmatrix} b \\ \dots \\ 0 \end{bmatrix} \left(f_x^T x_p + f_I x_I \right) \\ &= \bar{A}_m \bar{e} + g \bar{b} f^T x \end{aligned}$$

Clearly, it is now a well-known error equation, and

$$\bar{b} = \begin{bmatrix} b \\ \dots \\ 0 \end{bmatrix} \quad \textit{is known}$$

as b is known.

The adaptive law should be

$$\bar{A}_m^T \bar{P} + \bar{P} \bar{A}_m = -\bar{Q}$$

$$1\text{st } \textit{choose} \quad \bar{Q}, \ 2\textit{nd} \ \ \textit{calculate} \quad \bar{P}$$

$$\begin{bmatrix} \dot{q}_x \\ \dot{q}_I \end{bmatrix} = -\operatorname{sgn}(g) \Gamma \begin{bmatrix} x_p \\ x_I \end{bmatrix} \bar{e}^T \bar{P} \bar{b}$$

Adaptive control with all state-variables measurable: Incorporation of integral control

Example

Consider a plant

$$\dot{y}_p = a_p y_p + gu$$

To incorporate integral control, we know we must augment the state with

$$\dot{x}_I = y_p - r$$

The state of the new "plant" (including the augmented variable) is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} a_p & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + g \begin{bmatrix} 1 \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ -1 \end{bmatrix} r$$

with $x_1 \equiv y_p$ and $x_2 = x_I$

The feedback control is of the form

$$u(t) = q_1(t)x_1(t) + q_2(t)x_2(t)$$

(Refer to earlier notes.)

Consider constant gain feedback first. The plant with feedback is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} a_p + gq_1 & gq_2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} r$$

This means that the reference model used must have the form

$$\begin{bmatrix} \dot{x}_{1m} \\ \dot{x}_{2m} \end{bmatrix} = \begin{bmatrix} a_{1m} & a_{2m} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_{1m} \\ x_{2m} \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} r$$

- This is the only class of reference model that can be matched -------“matching conditions”.
- The analysis then proceeds in the normal way.

Exercise 2

Consider the plant $y = G_{yu}u$

where

$$G_{yu}(s) = \frac{g}{s^2 + a_1 s + a_2}$$

with $a_1 = 0$, $a_2 = 1$, y and $\dot{y}$ measurable.

Design an adaptive controller incorporating integral action.

Explain carefully how you choose the reference model to track.

In the limit as $t \to \infty$, what is the equivalent relationship between y(t) and r(t)?

(Express this in transfer function form.)

Exercise 3: Proportional plus integral adaptation

Consider the plant

$$\dot{y} = a_p y + u$$

and the reference model

$$\dot{y}_m = a_m y_m + r$$

Then, the adaptive controller

$$e = y - y_m$$
$$u = qy + r$$
$$\dot{q} = -gye \qquad g > 0$$

will result in asymptotic tracking.

Astrom (pp.136,137) has suggested an adaptive law of the form

$$q(t) = -g_1 ye - g_2 \int_0^t y(t)e(t)dt$$

Using the methods we have presented, analysis this adaptive law.

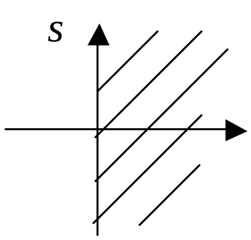
Positive Real Transfer Function

Definition: A rational transfer function H with real coefficients is positive real (PR) if *Re H(s)>0* for *Re s>0*.

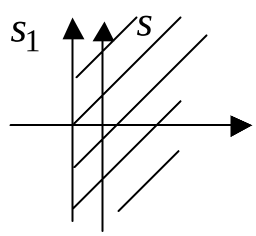
A transfer function H is strictly positive real (SPR) if H(s-ε) is positive real for some real ε>0.

Astrom¼p129

Narendra ¼p65



PR



SPR

$s = s_1 + e$

Kalman Yakuborich Lemma, <u>Lemma 2.3</u> (Narendra ¼ p66)

Given a scalar $g \geq 0$, a vector h, an asymptotically stable matrix *A*, a vector *b* such that *(A, b)* is controllable, and a positive definite matrix *L*, there exist a scalar ε>0, a vector q and a symmetric positive-definite matrix *P* satisfying $$A^T P + PA = -qq^T - eL$$ $$Pb - h = \sqrt{g}\, q$$ if and only if $H(s) \overset{\Delta}{=} \frac{1}{2} g + h^T (sI - A)^{-1} b$ is SPR.	$\dot{x} = Ax + bu$ $y = hx + \frac{1}{2} gu$ $(A, b, c, d) = (A, b, h, \frac{1}{2} g)$

Lemma 4-2 (Astrom p129) is a special case of preceeding with

$$g = 0$$

Positive real functions have been used extensively in network analysis.

The use of the concept in adaptive control is relatively recent.

A passive network (consisting only of inductance, resistance and capacitance) is positive real. If in addition, the network is dissipative due to the presence of resistors, then the impedance function is strictly positive real.

Viewed from realization, any PR (SPR) impedance function can be realized by a passive (dissipative) network.

(Narendra pp.63)

H : is said to be passive if $\exists\, b \in R$ such that

$$< Hx, x >_t = \int_0^t (Hx)^T x dt \geq b, \forall t \in R^+$$

$$(< y, x >_t = \int_0^t y^T x dt \geq b, \forall t \in R^+)$$

When H is a linear invariant operator, if H is passive, then H is PR.

(Narendra pp.71)

Continuous-time adaptive control using only input-output measurements

- Theory of SPR function will be used to prove stability.

References: Narendra……Chapter 5
Astrom………Section 4.5

Consider a system described by

$$R_p(p)y(t) = k_p Z_p(p)u(t) \qquad (1)$$

where

$$p \equiv \frac{d}{dt}$$

$$R_p(p) = p^n + a_1 p^{n-1} + \cdots + a_n$$
$$Z_p(p) = p^m + b_1 p^{m-1} + \cdots + b_m$$

R_p *is monic (i.e. leading coeff=1)*

$\deg(R_p) = n, \qquad \deg(Z_p) = m$

Define "relative degree" n^*

$$n^* \overset{\Delta}{=} n - m \qquad \textit{(i.e. excess poles over zeros.)}$$

Consider "Diophantine" identity

$$T(p)R_m(p) = R_p(p)E(p) + F(p)$$

$\left.\begin{array}{r}\deg(T) = n \\ \deg(R_m) = n^*\end{array}\right\}$	*Design polynomials, chosen to be stable, and monic*

For given plant, i.e. R_p, and specified T and R_m

E and F are unique, and

$$\deg(E) = n^*, \quad E \quad monic, \quad \deg(F) = n - 1$$

$$T(p) = p^n + t_1 p^{n-1} + ... + t_{n-1} p + t_n$$

Plant: $R_p(p)y = k_p Z_p(p)u$

$$ER_p y = k_p EZ_p u$$

i.e.

$$TR_m y = Fy + k_p EZ_p u = k_p(\overline{F}y + \overline{G}u)$$

$$\left(\overline{F} = \frac{F}{k_p}, \overline{G} = EZ_p \right)$$

$$R_m y = k_p\left(\frac{\overline{F}}{T}y + \frac{\overline{G}}{T}u\right) \qquad (5)$$

Note that (a) Z_p monic, and E monic, $\therefore$ $\overline{G}$ monic;

(b) $\deg(\overline{G}) = \deg(Z_p) + \deg(E) = n$

Accordingly, we have

$$\frac{\overline{G}}{T} = 1 + \frac{G_1}{T}$$

where

$$G_1(p) = g_1 p^{n-1} + g_2 p^{n-2} + \cdots + g_n$$

Similiarly, we can write

$$\overline{F}(p) = f_1 p^{n-1} + f_2 p^{n-2} + \cdots + f_n$$

Equation (5) then becomes

$$R_m y = k_p (\frac{\bar{F}}{T} y + \frac{G_1}{T} u + u) \qquad (7)$$

Define $y^{f_1} = \frac{1}{T} y$, $u^{f_1} = \frac{1}{T} u$

$$\bar{q}^* = \left[- f_1, - f_2, \cdots - f_n, -g_1, -g_2, \cdots - g_n, k^*\right]^T$$

$$\bar{w} = \left[p^{n-1} y^{f_1}, p^{n-2} y^{f_1}, \cdots, y^{f_1}, p^{n-1} u^{f_1}, p^{n-2} u^{f_1}, \cdots, u^{f_1}, r\right]^T$$

and r(t) is the set-point signal.

Consider

$$\begin{aligned} u(t) &= -\frac{\bar{F}}{T} y - \frac{G_1}{T} u + k^* r = -\bar{F} y^{f_1} - G_1 u^{f_1} + k^* r \\ &= \bar{q}^{*T} \bar{w} = q_u^{*T} w_u + q_y^{*T} w_y + k^* r \end{aligned} \qquad (8)$$

Equation (7) becomes

$$\begin{aligned} R_m y &= k_p k^* r \\ &= k_m r \qquad for \quad k_p k^* = k_m \end{aligned}$$

The choice of the input (8) ensure that the plant with feedback becomes

$$R_m(p)y(t) = k_m r(t)$$

In other words, the plant output will track a reference output (reference model) given by

$$R_m(p)y_m(t) = k_m r(t) \quad \Rightarrow \quad \frac{Y_m}{R} = \frac{k_m}{R_m(s)}$$

Remember,

$$\deg(R_m) = n^*$$

R_m *is a design polynomial you choose.*

Example:

n^*	Choice
$n^* = 1,$	$R_m(s) = (s + a_m)$ $k_m = a_m;$
$n^* = 2$	$R_m(s) = s^2 + 2xw_n s + w_n^2$ $k_m = w_n^2$

However,

while we can construct the signals in $\overline{w}$ *,*

we do not know the desired controller parameters $\overline{q}^*$*.*

Try using

$$u(t) = \overline{q}(t)^T \overline{w}(t)$$

and adaptively determine $\overline{q}(t)$ *in some way.*

This is possible. Two separate cases:

(i) $n^* = 1$, *convergence proof easy (relatively)*

(ii) $n^* > 1$, *convergence proof possible but very difficult*

Recall equation (7)

$$\begin{aligned} R_m y &= k_p\left(\frac{\overline{F}}{T}y + \frac{G_1}{T}u + u\right) \\ &= k_p\left(\left[\frac{\overline{F}}{T}y + \frac{G_1}{T}u - k^* r\right] + k^* r + u\right) \\ &= k_p\left(-\overline{q}^{*T}\overline{w} + k^* r + u\right) \end{aligned}$$

Control law:

$$u(t) = \overline{q}(t)^T \overline{w}(t)$$

Define parameter error

$$\overline{f}(t) \stackrel{\Delta}{=} \overline{q}(t) - \overline{q}^*$$

Then above becomes

$$R_m y = k_p\left(\overline{f}(t)^T \overline{w}(t) + k^* r(t)\right)$$

Comparing with reference model

$$R_m y_m = k_m r = k_p k^* r$$

gives error equation

$$R_m e_1 = k_p \overline{f}(t)^T \overline{w}(t)$$

where $e_1 \stackrel{\Delta}{=} y - y_m$.

At this point, remembering $n^* = 1$ and $\deg(R_m) = n^*$

we have $R_m(s) = s + a_m$ (stable reference model)

It is straightforward to check that

$$\frac{y_m}{r} = W_m = \frac{k_m}{R_m(s)} = \frac{k_m}{s + a_m}; \qquad k_m > 0$$

is SPR.

We will use this property together with the Ralman-Yakubovich Lemma to prove stability and convergence of $e_1(t)$ to zero.

However, we need first a state realization which includes

$$w(t) \stackrel{\Delta}{=} \left[p^{n-1} y^{f_1}, p^{n-2} y^{f_1}, \ldots, y^{f_1}, p^{n-1} u^{f_1}, p^{n-2} u^{f_1}, \ldots, u^{f_1} \right]^T$$

as states (in order to prove boundedness of all state variables).

Proposition 1: The vector $w(t)$ is the state of a realization (nonminimal) of the plant

$$R_p y(t) = k_p Z_p u(t)$$

Proof:

Recall that, $T = p^n + t_1 p^{n-1} + \ldots + t_{n-1} p + t_n$

$$Tu^{f_1} = u \qquad (1)$$

and clearly

$$R_p y^{f_1} = k_p Z_p u^{f_1} \qquad (2)$$

Further, since $\deg(T) = \deg(R_p) = n$, clearly

$$\begin{aligned} y &= Ty^{f1} = Ty^{f1} + 0 \\ &= (T - R_p)y^{f1} + k_p Z_p u^{f1} \end{aligned} \qquad (3)$$

Equations (1), (2) and (3) form the basis for a 2n-dimensional state representation of the plant.

From (1), we have

$$\frac{d}{dt}\begin{bmatrix} u^{f_1} \\ pu^{f_1} \\ \vdots \\ p^{n-1}u^{f_1} \end{bmatrix} = A_T \begin{bmatrix} u^{f_1} \\ pu^{f_1} \\ \vdots \\ p^{n-1}u^{f_1} \end{bmatrix} + b_n u$$

where

$$A_T = \begin{bmatrix} \mathbf{0} & \mathbf{I}_{(n-1)\times(n-1)} & & \\ -t_n & -t_{n-1} & \cdots & -t_1 \end{bmatrix} \text{ and } b_n = \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix}$$

This can be combined with (2) and (3) to give

$$\frac{d}{dt}w(t)=\begin{bmatrix}\mathbf{A}_{R_p} & \mathbf{A}_{k_pZ_p}\\ \mathbf{0}_{n\times n} & \mathbf{A}_T\end{bmatrix}w(t)+\begin{bmatrix}\mathbf{0}\\ b_n\end{bmatrix}u(t)$$

$$y(t)=\left[t_n-a_n,t_{n-1}-a_{n-1},\cdots,t_1-a_1,0,\cdots,0,k_pb_m,\cdots,k_pb_1\right]w(t)$$

The forms of the $n\times n$ matrices $\mathbf{A}_{R_p}$ and $\mathbf{A}_{k_pZ_p}$ should be obvious. The above is a state realization of the plant with $w(t)$ as the state.

QED

Proposition 2: For the state realization given previously, there exist controller gains q^* such that the closed-loop poles are at the roots of the 2n degree polynomial $TR_m Z_p$.

Proof:

Most of the materials have been introduced already. It only remains to put it into a formal basis.

Recall the Diophantine identity

$$TR_m = R_p E + F$$

and the plant equation

$$R_p y = k_p Z_p u$$

From earlier, recall that our control law was

$$u = q^{*T} w + k^* r \qquad (1)$$

and it is equivalent to

$$\bar{F} y^{f_1} + \bar{G} u^{f_1} = k^* r \qquad (2)$$

wih $k_p \bar{F} = F$.

[Why?

p.44, $u(t) = q^{*T} w + k^* r = -\frac{\bar{F}}{T} y - \frac{G_1}{T} u + k^* r \qquad \Rightarrow$

$$\frac{\bar{F}}{T} y + \left(1 + \frac{G_1}{T}\right) u = k^* r \qquad \Rightarrow \qquad \frac{\bar{F}}{T} y + \frac{\bar{G}}{T} u = k^* r$$

]

Observe that

$$\begin{aligned} TR_m Z_p &= R_p E Z_p + F Z_p \\ &= R_p \overline{G} + Z_p F \\ &= R_p \overline{G} + k_p Z_p \overline{F} \end{aligned} \qquad (3)$$

If the control law (1) or equivalent (2) is applied to the plant, we have:

Plant: $R_p y = k_p Z_p u$

$$R_p y^{f_1} = k_p Z_p u^{f_1}$$

From (2)

$$\begin{aligned} R_p \overline{G} y^{f_1} &= k_p Z_p \overline{G} u^{f_1} \\ &= k_p Z_p \left\{ k^* r - \overline{F} y^{f_1} \right\} \end{aligned}$$

or

$$\left(R_p \overline{G} + k_p Z_p \overline{F} \right) y^{f_1} = k_p k^* Z_p r$$

From (3) $TR_m Z_p y^{f_1} = k_m Z_p r$

i.e. the closed loop poles are at $TR_m Z_p$.

In addition, if T and Z_p are stable, then the closed-loop transfer function is

$$R_m y = k_m r$$

QED

Corollary:

The reference model

$$R_m(p)y_m(t) = k_m r(t) \qquad i.e., \qquad \frac{Y_m}{R} = \frac{k_m}{R_m(s)}$$

admits a 2n dimensional state-representation

$$\dot{w}_m = A_m w_m + b_m r$$
$$y_m = c_m^T w_m$$

Proof:

Straightforward by combining previous two propositions.

Note that:

$R_m y_m = k_m r$	$\Rightarrow$	$W_m = \frac{y_m}{r} = \frac{k_m}{R_m}$
$\dot{w}_m = A_m w_m + b_m r$ $y_m = c_m^T w_m$	$\Rightarrow$	$W_m = \frac{y_m}{r} = k_m^T (sI - A_m)^{-1} b_m$

$\Rightarrow$

$$W_m = k_m^T (sI - A_m)^{-1} b_m = \frac{k_m}{R_m}$$

Stability Analysis

Use the 2n-dimensional state-representation

$$\dot{w} = A_p w + b_p u$$
$$y = c_p^T w$$

Consider

$$u = \bar{q}(t)^T \bar{w} = q(t)^T w + k(t) r$$
$$= \{q^* + f(t)\}^T w + \{k^* + f_k\} r$$

Then, C.L.

$$\dot{w} = \left(A_p + b_p q^{*T} \right) w + b_p \left(f^T w + f_k r \right) + k^* b_p r$$
$$y = c_p^T w$$

For the exact values q^*, k^*, the plant with feedback matches the reference model,

$$A_p + b_p q^{*T} = A_m; \qquad k^* b_p = b_m$$

and $c_p = c_m$

Consider now the state error

$$e \stackrel{\Delta}{=} w - w_m, \qquad e_1 = y - y_m$$

Then

$$\dot{e} = A_m e + \frac{1}{k^*} b_m \bar{f}^T \bar{w}$$

$$e_1 = c_m^T e$$

$$e_1 = c_m^T [sI - A_m]^{-1} \frac{b_m}{k^*} \bar{f}^T \bar{w}$$

Noting that (A_m, b_m, c_m) is a state representation of the reference model, we have

$$c_m^T (sI - A_m)^{-1} b_m = \frac{k_m}{R_m}$$

This is strictly positive real, as is $c_m^T (sI - A_m)^{-1} \frac{1}{|k^*|} b_m = \frac{k_m}{|k^*| R_m}$.

Consider Lyapunov function condidate

$$V(e, f) = e^T P e + \bar{f}^T \Gamma^{-1} \bar{f}$$

where

$$A_m^T P + P A_m = -Q$$

Then

$$\dot{V} = -e^T Q e + 2 e^T P \frac{1}{k^*} b_m \bar{f}^T \bar{w} + 2 \bar{f}^T \Gamma^{-1} \dot{\bar{f}}$$

Since $\left(A_m, \dfrac{1}{|k^*|} b_m, c_m \right)$ is SPR, KY-Lemma ensures that

$$P \frac{1}{|k^*|} b_m = c_m$$

which means that

$$\begin{aligned} \dot{V} &= -e^T Qe + 2e^T c_m \operatorname{sgn}(k^*) \bar{f}^T \bar{w} + 2\bar{f}^T \Gamma^{-1} \dot{\bar{f}} \\ &= -e^T Qe + 2e_1^T \operatorname{sgn}(k^*) \bar{f}^T \bar{w} + 2\bar{f}^T \Gamma^{-1} \dot{\bar{f}} \end{aligned}$$

If

$$\dot{\bar{q}} = \dot{\bar{f}} = -\operatorname{sgn}(k^*) \Gamma \bar{w} e_1 = -\operatorname{sgn}(k_p) \Gamma \bar{w} e_1 \quad \text{as} \quad k_m > 0$$

we have

$$\dot{V} = -e^T Qe \le 0$$

Thus

(a) $\|e\|$, $\|\bar{f}\|$ (hence $\|w\|$, $\|\bar{q}\|$) are bounded;

(b) $\dot{e}$ bounded, $\int_0^\infty e(t)^T Qe(t) dt \le c_1$

$$\therefore \lim_{t \to \infty} e(t) = 0$$

$$\Rightarrow \lim_{t \to \infty} e_1(t) = 0$$

Case $n^* = 1$, Summary

Plant:

$$R_p y = k_p Z_p u$$

Control:

$$y^{f_1} = \frac{1}{T} y, \qquad u^{f_1} = \frac{1}{T} u$$

$$\overline{w}(t) \overset{\Delta}{=} \left[p^{n-1} y^{f_1}, p^{n-2} y^{f_1}, \cdots, y^{f_1}, p^{n-1} u^{f_1}, p^{n-2} u^{f_1}, \cdots, u^{f_1}, r \right]^T$$

$$u(t) = \overline{q}(t)^T \overline{w}(t)$$

Adaptive law

$$\dot{\overline{q}} = -\mathrm{sgn}(k_p) \Gamma w e_1$$
$$e_1 = y - y_m$$

Result: If

(a) order of R_p, n, is known;

(b) Z_p is stable polynomial;

(c) sgn(k_p) is known;

then the above system leads to y(t), u(t), $\overline{q}(t)$ bounded, and

$$\lim_{t \to \infty} \left(y(t) - y_m(t) \right) = 0$$